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New generalizations of all the known integrable problems in rigid-body dynamics

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Abstract. In the present paper we consider a quite general problem of motion of a rigid body under the action of an axially symmetric combination of potential and gyroscopic forces. Special versions of this problem are the classical problems of motion of a heavy body, gyrostat or a body in a liquid. Another example is the problem of motion of a heavy, magnetized and electrically charged body under the combined action of gravitational, magnetic electric and Lorentz forces.

We introduce a method, based on the invariance of the equations of motion under certain transformations. This method enables a systematic construction of several integrable problems generalizing all the known general (for arbitrary initial conditions) and conditional (on a single level of the cyclic integral) integrable problems as well as particular solutions. Five general integrable cases are constructed, that generalize known ones by the inclusion of one to four additional free parameters. Fifteen conditional integrable cases generalizing all the known general and conditional cases and containing an additional arbitrary function are introduced. The latter type of generalization can be applied to any one of a large collection of particular solutions known in many versions of the problem considered. The method also enables the construction of the explicit solutions of the equations of motion in the generalized cases from those of the original cases. Detailed physical interpretation is given for some cases.

1. Introduction

The subject of rigid-body dynamics is one of the oldest subjects that have been intensively studied for a long time. It became clear after the works of Kovalevskaya, Liouville and others that the problem of motion of a heavy rigid body about a fixed point can be reduced to quadratures only in the three cases of Euler, Lagrange and Kovalevskaya (e.g. [10, 51]). When certain other more general problems were considered, some integrable problems were found that generalize those three cases. An example is the problem of motion of a body in a liquid (see e.g. [7,25]), admitting six integrable cases [5]. Recently, several methods have been developed and applied to prove the nonintegrability of certain limited versions of the above problems and thus validating the rarity of integrable problems in dynamics (e.g. [52–54] and references cited therein).

The question arises whether it is possible to determine the moments of forces that lead to integrable problems when applied to the body. As there is no available method capable of dealing with this question, it acquires great importance to construct, classify and tabulate as much integrable cases as possible and in their most general setting. A natural way is to find as wide as possible generalizations of the known integrable problems.

† Dedicated to the memory of Professor V G Dumin, my teacher and friend.

In a recent short paper [1] we have pointed out six new general integrable cases that generalize six of the seven known general integrable cases in rigid-body dynamics by adding to their structure a set of arbitrary parameters. In the present paper we present the method that has led to five of those generalized cases. We show that this method enables systematic construction of several conditional integrable problems generalizing all the known general and conditional integrable problems, by the inclusion of an arbitrary function in their structure. The same generalization applies for all known particular solutions of the integrable and nonintegrable problems. Moreover, this method allows the construction of explicit solution of each generalized case in terms of the solution of the original case whenever the latter is known. It also enables us to deduce certain qualitative properties of motion in the generalized cases from those of the original ones.

Consider a rigid body in motion about its fixed point O. Let $OXYZ$ and $Oxyz$ be two Cartesian coordinate systems, fixed in space and in the body, respectively. Also let $\boldsymbol{\omega} = (p, q, r)$ be the angular velocity of the body and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ be the unit vector in the direction of the Z-axis, both being referred to as the body system which we take as the system of principal axes of inertia.

Those variables can be expressed in terms of Euler's angles: ψ the angle of precession about the Z-axis, θ the angle of nutation (between the z- and Z-axes) and the angle of proper rotation φ about the z-axis. They have the form

$$\gamma_1 = \sin\theta\sin\varphi \qquad \gamma_2 = \sin\theta\cos\varphi \qquad \gamma_3 = \cos\theta$$
$$p = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi \qquad q = \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi \qquad r = \dot{\psi}\cos\theta + \dot{\varphi}.$$

The problem considered here is the general problem of motion of a rigid body about a fixed point under the action of a combination of a conservative axisymmetric around the Z-axis potential and gyroscopic forces, described by the Lagrangian [4]:

$$L = \tfrac{1}{2}\boldsymbol{\omega}\boldsymbol{I}\cdot\boldsymbol{\omega} + \boldsymbol{l}\cdot\boldsymbol{\omega} - V \tag{1}$$

where $\boldsymbol{I} = \operatorname{diag}(A, B, C)$ is the inertia matrix of the body. The potential V and the vector $\boldsymbol{l}$ depend only on the Poisson variables $\gamma_1, \gamma_2, \gamma_3$.

The equations of motion for this problem can be written in the Euler–Poisson form [4]:

$$\begin{aligned} &\dot{\boldsymbol{\omega}}\boldsymbol{I} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\boldsymbol{I} + \boldsymbol{\mu}) = \boldsymbol{\gamma}\times\frac{\partial V}{\partial\boldsymbol{\gamma}} \\ &\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} = \mathbf{0} \end{aligned} \tag{2}$$

where

$$\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) = \frac{\partial}{\partial\boldsymbol{\gamma}}(\boldsymbol{l}\cdot\boldsymbol{\gamma}) - \left(\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot\boldsymbol{l}\right)\boldsymbol{\gamma}. \tag{3}$$

As was shown in [4], potential terms of equations (2) can be interpreted in most cases in terms of three classical interactions: gravitational, electric and magnetic. Gyroscopic terms can be accounted for by attaching rotors to the body and adding Lorentz forces.

Equations (2) admit three general first integrals: (a) Jacobi's integral I_1 [1]. This coincides with the integral of total energy when the moments of gyroscopic forces vanish, i.e. when $\boldsymbol{\mu} = \mathbf{0}$. (b) The geometric integral $|\boldsymbol{\gamma}|^2 = 1$. (c) An integral linear in the components of angular velocity corresponding to the cyclic angle of precession around the axis of symmetry of the fields:

$$I_3 = (\boldsymbol{\omega}\boldsymbol{I} + \boldsymbol{l})\cdot\boldsymbol{\gamma} = \text{const} = f. \tag{4}$$

This integral is usually called the area's integral. The system (2) will be integrable in the sense of Liouville, as well as in the sense of Jacobi, whenever the integral I_4 exists and is functionally independent of I_1, I_3 [1].

2. A transformation of the equations of motion

In [3–5] we have applied the transformation $\boldsymbol{\omega} = \boldsymbol{\omega}' + \nu\boldsymbol{\gamma}$, where ν is a constant, to the system (2) and used it to generate integrable cases containing ν as a parameter. In [6], on the basis of the properties of the equation of motion maximally reduced to a single equation of the second order, we have indicated the possibility to use the same transformation with ν as a function of $\boldsymbol{\gamma}$. Here we deal with this idea in detail. In fact, the substitution

$$\boldsymbol{\omega} = \boldsymbol{\omega}' + \nu\boldsymbol{\gamma} \qquad \nu = \nu(\gamma_1, \gamma_2, \gamma_3) \tag{5}$$

leaves invariant the form of the Poisson equation in (2), transforming it to

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}' \times \boldsymbol{\gamma} = \mathbf{0} \tag{6}$$

while (4) takes the form

$$I_3 = (\boldsymbol{\omega}'\boldsymbol{I} + \boldsymbol{l} + \nu\boldsymbol{\gamma}\boldsymbol{I}) \cdot \boldsymbol{\gamma} = f. \tag{7}$$

Substituting in the Eulerian part of the equations of motion, using (6) and rearranging terms, we get:

$$\begin{aligned}\dot{\boldsymbol{\omega}}'\boldsymbol{I} + \boldsymbol{\omega}' \times \left(\boldsymbol{\omega}'\boldsymbol{I} + \boldsymbol{\mu} + 2\nu\boldsymbol{\gamma}\boldsymbol{I} - \nu(\operatorname{tr}\boldsymbol{I})\boldsymbol{\gamma} + \boldsymbol{\gamma}\boldsymbol{I}\cdot\boldsymbol{\gamma}\frac{\partial\nu}{\partial\boldsymbol{\gamma}} - \left(\boldsymbol{\gamma}\boldsymbol{I}\cdot\frac{\partial\nu}{\partial\boldsymbol{\gamma}}\right)\boldsymbol{\gamma}\right)\\ = \boldsymbol{\gamma} \times \left(\frac{\partial V}{\partial\boldsymbol{\gamma}} - \nu\boldsymbol{\mu} - \nu^2\boldsymbol{\gamma}\boldsymbol{I} + (\boldsymbol{\omega}'\boldsymbol{I}\cdot\boldsymbol{\gamma})\frac{\partial\nu}{\partial\boldsymbol{\gamma}}\right).\end{aligned} \tag{8}$$

On the level $I_3 = f$ (say), we substitute $\boldsymbol{\omega}'\boldsymbol{I}\cdot\boldsymbol{\gamma}$ from (7) and after some manipulations write the equations of motion in the final form:

$$\begin{aligned}&\dot{\boldsymbol{\omega}}'\boldsymbol{I} + \boldsymbol{\omega}' \times (\boldsymbol{\omega}'\boldsymbol{I} + \boldsymbol{\mu}') = \boldsymbol{\gamma} \times \frac{\partial V'}{\partial\boldsymbol{\gamma}}\\ &\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}' \times \boldsymbol{\gamma} = \mathbf{0}\end{aligned} \tag{9}$$

where

$$\begin{aligned}\boldsymbol{\mu}' &= \boldsymbol{\mu} + \frac{\partial}{\partial\boldsymbol{\gamma}}(\nu\boldsymbol{\gamma}\boldsymbol{I}\cdot\boldsymbol{\gamma}) - \left[\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot(\nu\boldsymbol{\gamma}\boldsymbol{I})\right]\boldsymbol{\gamma}\\ &\equiv \boldsymbol{\mu} - 2\nu\boldsymbol{\gamma}\bar{\boldsymbol{I}} + \boldsymbol{\gamma}\boldsymbol{I} \times \left(\frac{\partial\nu}{\partial\boldsymbol{\gamma}} \times \boldsymbol{\gamma}\right)\\ V' &= V + \nu(f - \boldsymbol{l}\cdot\boldsymbol{\gamma}) - \tfrac{1}{2}\nu^2\boldsymbol{\gamma}\boldsymbol{I}\cdot\boldsymbol{\gamma}\end{aligned} \tag{10}$$

and $\bar{\boldsymbol{I}} = \frac{1}{2}\operatorname{tr}(\boldsymbol{I})\boldsymbol{\delta} - \boldsymbol{I}$. From the first of equations (10) and comparing with (3), we can also write the transformation law for the vector $\boldsymbol{l}$ in (1)

$$\boldsymbol{l}' = \boldsymbol{l} + \nu\boldsymbol{\gamma}\boldsymbol{I}. \tag{11}$$

Thus, the transformation (5) preserves the form of the equations of motion on a fixed level of I_3, changing only V, $\boldsymbol{\mu}$ (or $\boldsymbol{l}$) to V', $\boldsymbol{\mu}'$ (or $\boldsymbol{l}'$). The value f of I_3 enters in the potential V' as a parameter. The solution of the transformed equations of motion (10) can be obtained from that of (2) through the substitution (5).

From the form of (5), the transformed system (10) can be understood as describing the motion of the same body as in (2) referring it to a noninertial frame moving with the position-dependent angular velocity ν. The new terms that appeared in the transformed system are the inertial forces due to the rotation of the frame.

There is a different and more useful way of looking at (10). We shall make use of the situation that the transformation preserves the form of the equations of motion to understand the transformed equations on their own as describing the motion of a second body in the inertial frame under the forces determined by V', $\boldsymbol{\mu}'$. In other words, we consider the system (10) as formally generalizing (2) to which it reduces when $\nu = 0$. However, this will not prevent us from relating the solutions of the two systems by the (formal) transformation (5). This duality in interpretation is the key for understanding the present method. In the rest of this paper we will mostly regard the system (10) as a generalization of (2) rather than a transformed form of it.

From the above considerations we can readily deduce the following theorems connecting the solutions of the two systems.

Theorem 1. *If the first system, with $V, \boldsymbol{\mu}$, is integrable on a fixed level of the cyclic integral $I_3 = f$ with the fourth integral $I_4 = F(\boldsymbol{\omega}, \boldsymbol{\gamma})$, then for arbitrary $\nu(\boldsymbol{\gamma})$ the second system, with $V', \boldsymbol{\mu}'$ is integrable on the same level of the cyclic integral and its fourth integral is $F(\boldsymbol{\omega}' + \nu(\boldsymbol{\gamma})\boldsymbol{\gamma}, \boldsymbol{\gamma})$. The converse also holds.*

Theorem 2. *If $\{\boldsymbol{\omega} = \boldsymbol{\Omega}(t, \boldsymbol{\omega}^\circ, \boldsymbol{\gamma}^\circ)$, $\boldsymbol{\gamma} = \boldsymbol{\Gamma}(t, \boldsymbol{\omega}^\circ, \boldsymbol{\gamma}^\circ)\}$, is the general solution of the first system satisfying the arbitrary initial conditions $\{\boldsymbol{\omega} = \boldsymbol{\omega}^\circ, \boldsymbol{\gamma} = \boldsymbol{\gamma}^\circ\}$, then for arbitrary $\nu(\boldsymbol{\gamma})$ the solution of the second system, satisfying the initial conditions $\{\boldsymbol{\omega}' = \boldsymbol{\omega}'^\circ, \boldsymbol{\gamma} = \boldsymbol{\gamma}^\circ\}$ is*

$$\begin{aligned}\{\boldsymbol{\omega}' = \boldsymbol{\Omega}(t, \boldsymbol{\omega}'^\circ + \nu(\boldsymbol{\gamma}^\circ)\boldsymbol{\gamma}^\circ, \boldsymbol{\gamma}^\circ) - \nu(\boldsymbol{\Gamma}(t, \boldsymbol{\omega}'^\circ + \nu(\boldsymbol{\gamma}^\circ)\boldsymbol{\gamma}^\circ, \boldsymbol{\gamma}^\circ))\boldsymbol{\Gamma}(t, \boldsymbol{\omega}'^\circ + \nu(\boldsymbol{\gamma}^\circ)\boldsymbol{\gamma}^\circ, \boldsymbol{\gamma}^\circ) \\ \boldsymbol{\gamma} = \boldsymbol{\Gamma}(t, \boldsymbol{\omega}'^\circ + \nu(\boldsymbol{\gamma}^\circ)\boldsymbol{\gamma}^\circ, \boldsymbol{\gamma}^\circ)\}.\end{aligned} \tag{12}$$

Theorem 3. *If the first system admits any particular solution $\{\boldsymbol{\omega} = \boldsymbol{\Omega}(t), \boldsymbol{\gamma} = \boldsymbol{\Gamma}(t)\}$, then for arbitrary $\nu(\boldsymbol{\gamma})$ the second system admits the solution $\{\boldsymbol{\omega}' = \boldsymbol{\Omega}(t) - \nu(\boldsymbol{\Gamma}(t))\boldsymbol{\Gamma}(t), \boldsymbol{\gamma} = \boldsymbol{\Gamma}(t)\}$.*

Theorem 4. *The qualitative behaviour of the second system with respect to the part $\boldsymbol{\gamma}$ of variables is exactly the same as that of the first system.*

The last theorem follows from the fact that the solution of the second system for the Poisson variables $\boldsymbol{\gamma}$ is not affected by the function $\nu(\boldsymbol{\gamma})$.

In the following sections we discuss the consequences of the above theorems in application to known solvable problems of rigid-body dynamics. Theorem 1 ensures the integrability of the problem (9) on the level f of the cyclic integral and for arbitrary $\nu(\boldsymbol{\gamma})$ whenever the corresponding problem (2) is integrable, either for arbitrary initial conditions or only on a fixed level of the cyclic integral. Theorem 2 relates the explicit solutions of the two problems. Theorem 3 enables the generalization, by means of including the function ν, of particular solutions of (2), i.e. solutions not involving any arbitrary constants or involving a number of constants of motion less than needed to guarantee integrability. Theorem 4 will be used to deduce certain qualitative properties of the motion of the generalized system from those of the original system.

2.1. On general and conditional integrable cases in rigid-body dynamics

Throughout the present work we shall call a problem *general integrable* if I_4 exists for arbitrary initial conditions and *conditional integrable* if it admits a fourth integral I_4 only on a single level f of the cyclic integral I_3 but for all initial conditions compatible with that level. In both types of problems the solution can be reduced to quadratures through the application of Liouville's theorem or Jacobi's theorem to the reduced two-dimensional Hamiltonian system. It is thus sufficient to point out the fourth integral to ensure integrability in those cases. In some cases it becomes possible to construct a quantity constant only under other restrictions on the initial state of motion, which do not fit as conditions on the integral levels of I_3 and

I_1. Then one cannot apply Liouville's theorem to construct the solution and a procedure for accomplishing this task should be indicated separately. In such cases we deal with *particular solutions* of the problem.

Equations of motion of the form (2) cover a wide range of applications in rigid-body dynamics. Special cases are the classical problem of motion of a heavy body, its generalizations to the case of a gyrostat moving under potential and Lorentz forces. They also cover certain problems in which the body has no fixed point.

In tables 1 and 2 we shall give a list of all the known 15 general and conditional integrable cases in different versions of the general problem (2). As those cases are scattered in the literature, we summarize them in two tables: one for general and one for conditional cases. The cases that are special versions of another one containing a larger number of parameters are grouped under that one and conditions leading to special versions are given. In some of these cases a parameter n is present, which can be excluded by a rotation with constant angular velocity and thus it suffices to consider the special cases $n = 0$. The latter will be called the *basic cases*. For simplicity we provide the complementary integrals only for those basic cases. It should be noted that those tables do not cover particular solutions valid under any other restrictions on the initial motion. Several solutions of this type can be found in the literature dealing with various versions of the general problem discussed here. The list provided in [13], mainly concerning motion in the uniform gravity field, is now far from complete.

2.2. *The case of constant ν*

In that case the transformation (5) has the simple meaning of transforming the problem of motion to a new reference frame rotating with a uniform angular velocity ν around the space axis of symmetry. We are more interested in the following interpretation: let S and S' be two identical bodies subject in the inertial frame to forces characterized by the couples $(V, \boldsymbol{\mu})$ and $(V', \boldsymbol{\mu}')$, respectively. The motion of the body S with angular velocity $\boldsymbol{\omega}$ can be obtained in all its details from the motion of S' with angular velocity $\boldsymbol{\omega}'$ by rotating the whole picture with the speed n so that

$$\boldsymbol{\omega} = \boldsymbol{\omega}' + n\gamma. \tag{13}$$

On the other hand, in the case of constant n the constant term nf in the transformed potential can be discarded, so that (10) take the form

$$\begin{aligned} V' &= V - n\boldsymbol{l} \cdot \gamma - \tfrac{1}{2} n^2 \gamma \boldsymbol{I} \cdot \gamma \\ \boldsymbol{\mu}' &= \boldsymbol{\mu} - 2n\gamma \bar{\boldsymbol{I}} \qquad \text{or} \qquad (\boldsymbol{l}' = \boldsymbol{l} + n\gamma I). \end{aligned} \tag{14}$$

(1) The difference in the potentials of S' and S consists of two terms:

- (a) The term $-\frac{1}{2} n^2 \gamma \boldsymbol{I} \cdot \gamma$ is the resultant force of the centrifugal forces on the body due to the rotation. It also has the form of the approximate potential of the body under the action of a Newtonian centre of attraction (see, e.g., [10]).
- (b) The term $-n\boldsymbol{l} \cdot \gamma$ can be interpreted in various ways, depending on the form of the vector $\boldsymbol{l}$ that characterizes the gyroscopic forces. For instance, a constant gyrostatic moment $\boldsymbol{k}$ of a rotor in the body S induces on S' the potential term $-n\boldsymbol{k} \cdot \gamma$. That is the same as due to the effect of a uniform gravity field in the γ direction on a mass distribution whose centre lies on the axis of the rotor.

(2) The term $-2n\gamma \bar{\boldsymbol{I}}$ added to $\boldsymbol{\mu}$ (or $n\gamma \boldsymbol{I}$ added to $\boldsymbol{l}$) is due to the Coriolis forces on the body due to the uniform rotation. It can be interpreted as resulting from the Lorentz effect of a uniform magnetic field $\mathcal{B} = \gamma$ on a distribution of electric charges, fixed in the body. In

fact, the Lorentz force acting on the charge e at the point $\boldsymbol{r}$ is $\frac{e}{c}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times \mathcal{B}$, so that the total Lorentz moment on the body is

$$\sum \boldsymbol{r} \times \left(\frac{e}{c}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times \mathcal{B}\right) = \sum \boldsymbol{r} \times \left[\frac{e}{c}(\boldsymbol{\omega} \times r) \times \mathcal{B}\right]$$
$$= \boldsymbol{\omega} \times \frac{e}{c}\sum(\boldsymbol{r} \cdot \mathcal{B})\boldsymbol{r}. \tag{15}$$

Comparing to (2), we deduce that the change in $\boldsymbol{\mu}$, $\boldsymbol{\mu}_e$ (say) is given by

$$\boldsymbol{\mu}_e = -\sum \frac{e}{c}(\boldsymbol{r} \cdot \mathcal{B})\boldsymbol{r} = -\mathcal{B}\boldsymbol{J} \tag{16}$$

where $\boldsymbol{J}$ is a constant matrix, $\boldsymbol{J}_{ij} = -\sum \frac{e}{c}x_i x_j$. This makes in $\boldsymbol{l}$ an increment $\boldsymbol{l}_e = \frac{1}{2}\mathcal{B}\boldsymbol{I}_e = n\gamma\boldsymbol{I}$, where the inertia matrix of the electric distribution $\boldsymbol{I}_e = 2n\boldsymbol{I}$, i.e. must be proportional to the inertia matrix of the distribution of mass of the body.

Thus, one can always introduce the above combination of physical effects to the problem of motion without complicating its mathematics in any way. In particular, an integrable problem or any solution with the pair $(V, \boldsymbol{\mu})$ will lead to a family of integrable problems or solutions corresponding to $(V', \boldsymbol{\mu}')$ containing the additional parameter n.

In a few works, Grioli and other authors considered the motion of an electrically charged body, for which $\boldsymbol{I}_e$ is proportional to $\boldsymbol{I}$, under the action of Lorentz forces (e.g. [39–41]). In a much wider version we have studied the motion of a gyrostat under the action of a coaxial combination of: a uniform magnetic field and linear electric and gravitational fields [3]. It was noted in [5] that the mathematical formulation of the above version coincides with the new form of the problem of motion of a body in a liquid introduced in that work. This meant that each one of the six integrable cases of the latter problem gives a valid analogue in the problem of motion of a charged body.

3. Examples of generalized conditional cases and particular solutions

By theorem 1 in section 2 all the 15 cases listed in the two tables admit a generalization using the transformation (5) to conditional cases involving the arbitrary function $\nu(\gamma_1, \gamma_2, \gamma_3)$. The explicit solution of the equations of motion in each case can be obtained from the solution of the original case, if the last is known, by theorem 2. As an illustration we apply this procedure to some of the soluble cases in detail. All other cases in the two tables can be treated in the same way.

3.1. A conditional generalization of the Rubanovsky–Steklov general case

We take this case in its *basic* form (table 1: case 6):

$$\boldsymbol{\mu} = \boldsymbol{k} + a\gamma\boldsymbol{I}^{-1} \qquad V = 0 \tag{17}$$

for which

$$\begin{aligned} &\boldsymbol{l} = \boldsymbol{k} + \tfrac{1}{2}a[\mathrm{tr}(\boldsymbol{I}^{-1})\gamma - \gamma\boldsymbol{I}^{-1}] \\ &I_3 = (\boldsymbol{\omega}\boldsymbol{I} + \boldsymbol{l}) \cdot \gamma = f \\ &I_4 = \tfrac{1}{2}|\boldsymbol{\omega}\boldsymbol{I} + \boldsymbol{k}|^2 - a\boldsymbol{\omega} \cdot \gamma. \end{aligned} \tag{18}$$

Consider the system

$$\begin{aligned} &\dot{\boldsymbol{\omega}}\boldsymbol{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\boldsymbol{I} + \boldsymbol{\mu}) = \gamma \times \frac{\partial V}{\partial \gamma} \\ &\dot{\gamma} + \boldsymbol{\omega} \times \gamma = \boldsymbol{0} \end{aligned} \tag{19}$$

in which

$$\begin{aligned}\boldsymbol{\mu} &= \boldsymbol{k} + a\boldsymbol{\gamma}\boldsymbol{I}^{-1} + \frac{\partial}{\partial\boldsymbol{\gamma}}(\nu\boldsymbol{\gamma}\boldsymbol{I}\cdot\boldsymbol{\gamma}) - \left[\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot(\nu\boldsymbol{\gamma}\boldsymbol{I})\right]\boldsymbol{\gamma}\\ V &= \beta\nu - \nu(\boldsymbol{k}\cdot\boldsymbol{\gamma} + \tfrac{1}{2}a[\mathrm{tr}(\boldsymbol{I}^{-1}) - \boldsymbol{\gamma}\boldsymbol{I}^{-1}\cdot\boldsymbol{\gamma}]) - \tfrac{1}{2}\nu^2\boldsymbol{\gamma}\boldsymbol{I}\cdot\boldsymbol{\gamma}\end{aligned} \tag{20}$$

where β is any given constant and ν is an arbitrary function in $\boldsymbol{\gamma}$. This system admits the linear integral

$$I_3 = \boldsymbol{\omega}\boldsymbol{I}\cdot\boldsymbol{\gamma} + \boldsymbol{k}\cdot\boldsymbol{\gamma} + \tfrac{1}{2}a[\mathrm{tr}(\boldsymbol{I}^{-1}) - \boldsymbol{\gamma}\boldsymbol{I}^{-1}\cdot\boldsymbol{\gamma}] + \nu\boldsymbol{\gamma}\boldsymbol{I}\cdot\boldsymbol{\gamma} = f \tag{21}$$

and on the level $I_3 = \beta$ the system becomes integrable with the complementary integral

$$I_4 = \tfrac{1}{2}|\boldsymbol{\omega}\boldsymbol{I} + \nu\boldsymbol{\gamma}\boldsymbol{I} + \boldsymbol{k}|^2 - a(\boldsymbol{\omega}\cdot\boldsymbol{\gamma} + \nu). \tag{22}$$

This result can be checked directly using the equations of motion (19), (20) and the integral (21) and without referring to the above constructions. In fact, one can show, using a computer algebra package, that

$$\frac{\mathrm{d}I_4}{\mathrm{d}t} = \frac{f-\beta}{ABC\gamma_3}P \tag{23}$$

where P is a polynomial in the components of $\boldsymbol{\omega}$ depending on $\boldsymbol{\gamma}$ and the parameters of the system.

One must note here that we have not made any statement about the integrability of the system (19), (20) for arbitrary initial conditions i.e. on arbitrary level of $I_3 = f \neq \beta$. The special case $\nu(\boldsymbol{\gamma}) = n$ gives the Rubanovsky–Steklov case which is a general one, since the constant term $\beta\nu$ in the potential is immaterial in that case.

The explicit solution of the system (19), (20) on the level $I_3 = \beta$ can be obtained from the knowledge of the solution of the basic case $\nu = 0$ (characterized by (17), (18) and generalizing Steklov's result only by the inclusion of the gyrostatic moment). This was achieved only in two particular cases:

(1) In Steklov's case $\boldsymbol{k} = \boldsymbol{0}$, by Kötter in terms of theta functions of two variables [44].
(2) In Joukovsky's case $a = 0$, the solution was obtained by Volterra [45] in terms of Weierstrass functions. An alternative solution in terms of Jacobi's elliptic functions was constructed by Wittenburg [46].

Despite the interest in applying methods of modern algebraic geometry (e.g. [55]), the general solution for the full basic case $a|\boldsymbol{k}| \neq 0$ was not considered.

3.2. A conditional case related to Lagrange's type

Consider the motion of the body whose matrix of inertia is $\mathrm{diag}(A, A, C)$, under the forces corresponding to $V = v(\gamma_3)$ and $\boldsymbol{l} = (\ell\gamma_1, \ell\gamma_2, l_3)$, where ℓ, l_3 depend on γ_3 only. This problem is integrable with two cyclic integrals [1, 16]. Consider the following system involving the four arbitrary functions $v(\gamma_3), \ell(\gamma_3), l_3(\gamma_3)$ and $\nu(\gamma_1, \gamma_2, \gamma_3)$:

$$\begin{aligned}V &= v(\gamma_3) + b\nu - \nu[(\gamma_1^2 + \gamma_2^2)\ell + l_3\gamma_3] - \tfrac{1}{2}\nu^2(A + (C-A)\gamma_3^2)\\ \boldsymbol{\mu} &= \frac{\partial}{\partial\boldsymbol{\gamma}}(\boldsymbol{l}\cdot\boldsymbol{\gamma}) - \left(\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot\boldsymbol{l}\right)\boldsymbol{\gamma}\end{aligned} \tag{24}$$

where $\boldsymbol{l} = ((\ell + A\nu)\gamma_1, (\ell + A\nu)\gamma_2, l_3 + C\nu\gamma_3)$. This is a system with one cyclic coordinate—the precession angle ψ—corresponding to the integral

$$I_3 = A(p\gamma_1 + q\gamma_2) + (Cr + l_3)\gamma_3 + (\gamma_1^2 + \gamma_2^2)\ell + \nu[A + (C-A)\gamma_3^2].$$

The proper rotation angle φ is no longer cyclic and the integral

$$I_4 = C(r + \nu\gamma_3) + l_3$$

is conditional on the level $I_3 = b$. In fact, one can easily find

$$\frac{\mathrm{d}I_4}{\mathrm{d}t} = (I_3 - b)\left(\gamma_2 \frac{\partial \nu}{\partial \gamma_1} - \gamma_1 \frac{\partial \nu}{\partial \gamma_2}\right). \tag{25}$$

This expression vanishes when $I_3 = b$. Although the system with (24) may not be integrable in general, it is integrable on the level $I_3 = b$.

3.3. Generalization of particular solutions

A large number of particular solutions of (2) is known. Most of them are concerned with the classical problems of heavy body, heavy gyrostat and the body in a liquid (e.g. [10,13]). Much fewer cases deal with more general versions of (2), e.g. [58]. However, a complete up-to-date review of those solutions does not exist at the present time. As an illustrative example, we apply the transformation method here to obtain a generalization of Grioli's regular precession.

On a dynamical basis Grioli established the possibility of a regular precession of the heavy rigid body about a nonvertical axis under certain conditions on the parameters of the body [42]. Gulyaev derived the full explicit solution of this case [43] (see also [10]). We present the necessary details in brief. The solution differs from that of Gulyaev only in that we have assigned a certain value for the initial time moment, so that the solution becomes more transparent.

Let the axes be arranged such that $A \geqslant B \geqslant C$. For

$$V' = a\gamma_1 + c\gamma_3 \qquad \boldsymbol{\mu}' = 0 \tag{26}$$

where $a\sqrt{B-C} = c\sqrt{A-B}$, the system of equations (9) admits a particular solution

$$\begin{aligned}
&p' = \frac{\Omega}{s}(a - c\cos(\Omega t)) \qquad q' = \Omega\sin(\Omega t) \qquad r' = \frac{\Omega}{s}(c + a\cos(\Omega t))\\
&\gamma_1 = -\frac{\Omega^2}{s^2}[Cc\cos(\Omega t) + (B-C)a\sin^2(\Omega t)]\\
&\gamma_2 = \frac{\Omega^2}{s^3}\sin(\Omega t)[(Aa^2 + Cc^2) - (A-C)ac\cos(\Omega t)]\\
&\gamma_3 = \frac{\Omega^2}{s^2}[Aa\cos(\Omega t) + (A-B)c\sin^2(\Omega t)]
\end{aligned} \tag{27}$$

where $s = \sqrt{a^2+c^2}$, $\Omega^2 = \frac{s}{\sqrt{(A-B+C)^2+(A-B)(B-C)}}$. This solution corresponds to a uniform precession of the body. The angular velocity $\boldsymbol{\omega}'$ can be written as the sum of two terms

$$\boldsymbol{\omega}' = \Omega\boldsymbol{\zeta} + \Omega\boldsymbol{\alpha} \tag{28}$$

where $\boldsymbol{\zeta}$, $\boldsymbol{\alpha}$ are two unit vectors: the first fixed in the body (orthogonal to a circular section of the inertia ellipsoid) and the second fixed in space [43]

$$\boldsymbol{\zeta} = \left(\frac{a}{s}, 0, \frac{c}{s}\right) \qquad \boldsymbol{\alpha} = \left(-\frac{c}{s}\cos(\Omega t), \sin(\Omega t), \frac{a}{s}\cos(\Omega t)\right). \tag{29}$$

Note that $\boldsymbol{\zeta}$ is orthogonal to $\boldsymbol{\alpha}$ and that $\boldsymbol{\alpha}$ is inclined to the upward vertical vector $\boldsymbol{\gamma}$ at a fixed angle δ,

$$\cos\delta = \frac{A-B+C}{\sqrt{(A-B+C)^2 + (A-B)(B-C)}}. \tag{30}$$

The body rotates with the uniform velocity Ω around the vector $\boldsymbol{\zeta}$ fixed in it, while that vector rotates with the same velocity Ω about the direction $\boldsymbol{\alpha}$ fixed in space.

We now consider another case of motion of the same body as above, but we will replace V', $\boldsymbol{\mu}'$ by

$$\begin{aligned} V &= a\gamma_1 + c\gamma_3 - \tfrac{1}{2}n^2(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) \\ \boldsymbol{\mu} &= n((B+C-A)\gamma_1, (C+A-B)\gamma_2, (A+B-C)\gamma_3). \end{aligned} \tag{31}$$

where, for simplicity, n is taken as a constant. It is easy to verify that applying the substitution $\boldsymbol{\omega} = \boldsymbol{\omega}' + n\boldsymbol{\gamma}$ transforms (31) into (26). Thus, the system with (31) admits a particular solution representing Grioli's precession uniformly rotated with speed n about the vertical. In this solution $\gamma_1, \gamma_2, \gamma_3$ are the same as in (27), while

$$\begin{aligned} p &= \frac{\Omega}{s}(a - c\cos(\Omega t)) - \frac{n\Omega^2}{s^2}[Cc\cos(\Omega t) + (B-C)a\sin^2(\Omega t)] \\ q &= \Omega\sin(\Omega t) + \frac{n\Omega^2}{s^3}\sin(\Omega t)[(Aa^2 + Cc^2) - (A-C)ac\cos(\Omega t)] \\ r &= \frac{\Omega}{s}(c + a\cos(\Omega t)) + \frac{n\Omega^2}{s^2}[Aa\cos(\Omega t) + (A-B)c\sin^2(\Omega t)]. \end{aligned} \tag{32}$$

This case is a non-trivial generalization of Grioli's result [42]. It admits two interpretations as a motion of a body in liquid [5] or a motion of a charged body under potential and Lorentz forces as described in section 2.2 above. It is noteworthy that this gives a new result in both interpretations.

The angular velocity $\boldsymbol{\omega} = \Omega\boldsymbol{\zeta} + (\Omega\boldsymbol{\alpha} + n\boldsymbol{\gamma})$ no longer has constant magnitude as was the case in Grioli's precession. The resulting motion is not a regular precession. Although $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ are periodic functions of time, the motion is not in general periodic in space for arbitrary values of n. However, if $\frac{n}{\Omega}$ is rational the body returns periodically to its initial position. As far as we know, such motions have not been considered previously. This solution can be generalized further by choosing ν as a function of $\boldsymbol{\gamma}$.

4. New general integrable cases

An exceptionally interesting situation is when the basic potential contains a combination of terms with arbitrary constant multipliers. Let, say,

$$V = v + \sum a_i v_i \tag{33}$$

where $v, \{v_i\}$ depend on $\boldsymbol{\gamma}$. If the problem of motion is integrable, then the complementary integral will probably depend on the set of constants a_i. If in the transformation (5) we choose $\nu = n + \sum n_i v_i$, where $\{n_i\}$ are free parameters, then in (10) the transformed potential we have $V' = v + \sum(a_i + n_i f)v_i + fn - \nu\boldsymbol{l}\cdot\boldsymbol{\gamma} - \frac{1}{2}\nu^2\boldsymbol{\gamma}\boldsymbol{I}\cdot\boldsymbol{\gamma}$. The constant term fn can be dropped out as it does not contribute to the equations of motion. Moreover, the constants a_i may be redefined to absorb the terms containing f. Thus, if we set $a_i + n_i f = b_i$, the potential V' will depend on the arbitrary constants b_i but no more on f. Meanwhile, the complementary integral will eventually depend on f, which can be substituted by the linear integral I_3. As there have been no restriction on f in this procedure, we get a general integrable case that contains the arbitrary parameters $n, \{n_i\}$ more than the original one. It should be noted that the solution of the generalized problem can be easily obtained from that of the basic one by applying the considerations of section 2.

The above analysis applies to the first five of the seven known integrable cases in which the potential has the required structure (see table 1). In our short paper [1] those generalized general integrable cases were presented without details of the method used. Expressions for V, $\boldsymbol{\mu}$ and the integrals of motion were provided in a final form suitable for direct verification of the constancy of the integrals by virtue of the equations of motion.

4.1. On explicit solution of the general integrable cases

In this subsection we discuss the explicit solution of the generalized cases in terms of time. Detailed equations of motion as well as the integrals of motion will not be reproduced here. For them the reader is referred to [1] (see also the minor correction in [2]).

(1) The first case was obtained by applying the transformation (5) with

$$\nu = n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) \tag{34}$$

to the Tisserand–Brun case [22,23] (in our terminology, this is the basic case for Clebsch's case, no 1 of table 1). The solution for this case was obtained, in the context of solving Kirchhoff's equations, by Kötter in terms of theta functions of two arguments [47]. The solution for the special case $f = 0$ was found earlier by Weber [50]. It is obvious that the explicit solution in the generalized case, constructed according to theorem 2 of section 2, will be expressed in the same class of functions.

(2) The second case is characterized by the choice

$$\nu = n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2 \tag{35}$$

applied to case 2 of table 1. The basic case $n = n_1 = n_2 = n_3 = 0$, is closely related to the other Clebsch's case of triaxial body. The solution of this case can be expressed in terms of theta functions of two variables [47,55] and so will be the present generalization.

(3) The third case is obtained from case 3 table 1 by the choice

$$\nu = n + n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3. \tag{36}$$

Lyapunov's case $s_1 = s_2 = s_3 = n_1 = n_2 = n_3 = 0$ [27] was solved by Kötter, as well as the related Steklov case, in terms of theta functions of two arguments [44, 55]. This solution will cover the case $s_1 = s_2 = s_3 = 0$ for arbitrary n_1, n_2 It is obvious that to express the solution in the most general case it suffices to obtain the solution for the basic case $n = n_1 = n_2 = n_3 = 0$, $s_1s_2s_3 \neq 0$. This has not been done until now.

(4) The fourth case is obtained from case 4 of table 1 with the choice

$$\nu = n + n_1\gamma_1 + n_2\gamma_2. \tag{37}$$

The present result generalizes Kovalevskaya's case by including four physically significant parameters k, n, n_1, n_2. It also generalizes some earlier results of the present author. For $n_1 = n_2 = 0$, we get the case of a body in a liquid [5] and if, moreover, $n = 0$, we get the case of a heavy gyrostat found in [19] (see also [20]).

The Euler–Poisson variables in Kovalevskaya's case were expressed by Kovalevskaya herself in terms of hyperelliptic functions of time [18]. The solution was somewhat simplified and systematized by Kötter [48, 49], (see also [10]). Explicit expressions for the six variables can also be found in [10, 51]. Many qualitative and global properties of motion are discussed in [52] The same problem was treated in a large number of recent works using methods of modern algebraic geometry and the inverse scattering method (e.g. [55, 56] and references cited therein). Of special interest is the work [57], which relates the Kovalevskaya case to a special version ($f = 0$) of Clebsch's case by means of a rational transformation and thus explicit solutions for the first can be obtained from that of the other. The same idea was realized for our generalization of Kovalevskaya's case to the gyrostat by Gavrilov, who related it to the full case of Clebsch ($f \neq 0$) solvable in theta functions of two arguments [53]. Thus, it becomes evident that the generalized case under discussion here is solved in terms of the same functions.

(5) The fifth case is obtained from case 5 of table 1 and the choice

$$\nu = n + n_1\gamma_1 + n_2\gamma_2 + \frac{N}{\sqrt{1-\gamma_3^2}}. \tag{38}$$

No attempt was made to obtain the general solution in the present case for $\varepsilon \neq 0$. For this it suffices to find the solution for the basic case $n = n_1 = n_2 = N = 0$, i.e. the case of [28].

4.2. Comment on general integrable cases

Thus, of all known general integrable cases remains without generalization only the case of a body in liquid found by Rubanovsky [26] that includes as special versions an earlier case due to Steklov [8] and the case of a gyrostat by inertia considered by Joukovsky [29] and Volterra [45]. The reason is obviously that the basic potential function $V = 0$ in that case, see (17), does not have the structure (33), and the method of the present section does not apply to it. We remind ourselves that the full Rubanovsky–Steklov case was generalized in a previous section by the introduction of an arbitrary function, but only as a conditional case on a fixed level of the cyclic integral.

4.3. Physical interpretation of the new cases

In all the above cases the precession angle ψ is a cyclic variable. All the forces acting on the body are symmetric about the Z-axis. Let there be the following combination of static external fields: a gravitational field with potential Φ_g, an electric field of potential Φe and a magnetic field $\mathcal{B}$ whose scalar potential is Φ. Note that the three potentials can depend only on Z and $X^2 + Y^2$. The potential of the body can be written as

$$V = \Sigma(m\Phi_g + e\Phi_e + \mathcal{B}\cdot\boldsymbol{\sigma}) \tag{39}$$

where m, e and $\boldsymbol{\sigma}$ are the mass, electric charge and the magnetic moment contained in the element of the body which at the current moment occupies the point $\boldsymbol{r}(X, Y, Z)$ of the inertial frame.

We first note that V in all the new cases (see [1]) has polynomial expressions in the components of $\boldsymbol{\gamma}$, except in one case, namely the fifth, which involves an algebraic singular term. Thus, in most cases we find that it is suitable to choose the three potentials Φ_g, Φ_e and Φ to be polynomials in X, Y, Z, subject to Laplace's equation and to the axial symmetry condition. Due to the abundance of physical parameters representing the three distributions and the coefficients of the three potentials, it should be easy to adjust those parameters to match the potential in each case and, moreover, in a variety of choices. We turn now to the less obvious possibility to obtain an adequate interpretation of gyroscopic forces as Lorentz forces.

The vector $\boldsymbol{\mu}$ can be expressed directly in terms of the constant gyrostatic moment $\boldsymbol{K}$ and the magnetic field

$$\boldsymbol{\mu} = \boldsymbol{K} - \Sigma\frac{e}{c}(\boldsymbol{r}\cdot\mathcal{B})\boldsymbol{r}. \tag{40}$$

In the case when the scalar potential of the external magnetic field can be expressed as a sum like

$$\Phi(X, Y, Z) = \Phi_1(X, Y, Z) + \cdots + \Phi_N(X, Y, Z) \tag{41}$$

of homogeneous harmonic polynomials up to the Nth degree, the formula (40) can be replaced by

$$\boldsymbol{\mu} = \boldsymbol{K} + \frac{e}{c}\sum_{s=1}^{N} s\Sigma\Phi_s(X, Y, Z)\boldsymbol{r}. \tag{42}$$

This results from applying Euler's theorem for homogeneous functions.

We will consider further only two cases, the third and fourth of [1], in which the components of $\boldsymbol{\mu}$ are of the second degree. In those cases we can take only two axisymmetric harmonics in (41), i.e.

$$\Phi(X, Y, Z) = a_1 Z + a_2(3Z^2 - r^2) \tag{43}$$

thus giving

$$\boldsymbol{\mu} = \boldsymbol{K} + \Sigma\frac{e}{c}[a_1 Z + 2a_2(3Z^2 - r^2)]\boldsymbol{r} \tag{44}$$

or, expressed in terms of the body system of coordinates,

$$\boldsymbol{\mu} = \boldsymbol{K} + \Sigma\frac{e}{c}[a_1\boldsymbol{r}\cdot\gamma + 2a_2(3(\boldsymbol{r}\cdot\gamma)^2 - r^2)]\boldsymbol{r}. \tag{45}$$

Finally, we can write

$$\begin{aligned}
\mu_1 &= 6a_2(I_{xxx}\gamma_1^2 + I_{xyy}\gamma_2^2 + I_{xzz}\gamma_3^2 + 2I_{xxy}\gamma_1\gamma_2 + 2I_{xxz}\gamma_1\gamma_3 + 2I_{xyz}\gamma_2\gamma_3)\\
&\qquad + a_1(I_{xx}\gamma_1 + I_{xy}\gamma_2 + I_{xz}\gamma_3) - 2a_2(I_{xxx} + I_{xyy} + I_{xzz}) + K_1\\
\mu_2 &= 6a_2(I_{xxy}\gamma_1^2 + I_{yyy}\gamma_2^2 + I_{yzz}\gamma_3^2 + 2I_{xyy}\gamma_1\gamma_2 + 2I_{xyz}\gamma_1\gamma_3 + 2I_{yyz}\gamma_2\gamma_3)\\
&\qquad + a_1(I_{xy}\gamma_1 + I_{yy}\gamma_2 + I_{yz}\gamma_3) - 2a_2(I_{xxy} + I_{yyy} + I_{yzz}) + K_2\\
\mu_3 &= 6a_2(I_{xxz}\gamma_1^2 + I_{yyz}\gamma_2^2 + I_{zzz}\gamma_3^2 + 2I_{xyz}\gamma_1\gamma_2 + 2I_{xzz}\gamma_1\gamma_3 + 2I_{yzz}\gamma_2\gamma_3)\\
&\qquad + a_1(I_{xz}\gamma_1 + I_{yz}\gamma_2 + I_{zz}\gamma_3) - 2a_2(I_{xxz} + I_{yyz} + I_{zzz}) + K_3
\end{aligned} \tag{46}$$

where, for example, $I_{xx} = \Sigma ex^2$, $I_{xyz} = \Sigma exyz$ and so forth are moments of the charge distribution of the second and third degrees.

It is not hard now to verify that the third case of [1] corresponds to the choice

$$\begin{aligned}
&I_{xy} = I_{xz} = I_{yz} = 0 \quad a_1 I_{xx} = a - An \quad a_1 I_{yy} = b - An \quad a_1 I_{zz} = c - An\\
&I_{xyz} = 0 \quad \frac{I_{xxx}}{3} = I_{xyy} = I_{xzz} = -\frac{An_1}{6a_2}\\
&I_{xxy} = \frac{I_{yyy}}{3} = I_{yzz} = -\frac{An_2}{6a_2} \quad I_{xxz} = I_{yyz} = \frac{I_{zzz}}{3} = -\frac{An_3}{6a_2}\\
&K_1 = \tfrac{1}{3}An_1 \quad K_2 = \tfrac{1}{3}An_2 \quad K_3 = \tfrac{1}{3}An_3.
\end{aligned} \tag{47}$$

Similarly, the fourth case is obtained by the choice

$$\begin{aligned}
&I_{xy} = I_{xz} = I_{yz} = 0 \quad a_1 I_{xx} = -Cn \quad a_1 I_{yy} = -Cn \quad a_1 I_{zz} = -3Cn\\
&I_{xxz} = I_{xyz} = I_{yyz} = I_{zzz} = 0\\
&\frac{I_{xxx}}{n_1} = \frac{I_{yyy}}{n_2} = -\frac{3C}{4a_2} \quad \frac{I_{xyy}}{n_1} = \frac{I_{xxy}}{n_2} = -\frac{C}{4a_2}\\
&\frac{I_{xzz}}{n_1} = \frac{I_{yzz}}{n_2} = -\frac{5C}{12a_2}\\
&K_1 = \tfrac{2}{3}Cn_1 \quad K_2 = \tfrac{2}{3}Cn_2 \quad K_3 = Ck.
\end{aligned} \tag{48}$$

In both cases, a certain constant constituent of $\boldsymbol{\mu}$ results from a gyrostatic moment $\boldsymbol{K}$ of a rotor and the rest from electromagnetic interaction. Bearing in mind that, unlike mass distribution, the charge distribution can have negative density, we have more freedom in satisfying conditions (47) or (48) by real physical situations. Similar interpretation can be given for the first and second cases of this section by including a fourth-degree harmonic in (43).

Table 1. General integrable cases.

Reference	Conditions	V	$\boldsymbol{\mu}$	I_4 (for $n=0$)	Special cases
1 Clebsch [21]	—	$(A\gamma_1^2+B\gamma_2^2+C\gamma_3^2)(b-\frac{1}{2}n^2)$	$n((A-B-C)\gamma_1,$ $(B-C-A)\gamma_2,$ $(C-A-B)\gamma_3)$	$\frac{1}{2}(A^2p^2+B^2q^2+C^2r^2)$ $-b(BC\gamma_1^2+CA\gamma_2^2+AB\gamma_3^2)$	Tisserand ($n=0$) [22] Brun ($n=0$) [23] Euler ($n=a=0$) [11]
2 Clebsch [21]	$A=B=C$	$c_1\gamma_1^2+c_2\gamma_2^2+c_3\gamma_3^2$	$-An\boldsymbol{\gamma}$	$A(c_1p^2+c_2q^2+c_3r^2)$ $-(c_2c_3\gamma_1^2+c_3c_1\gamma_2^2+c_1c_2\gamma_3^2)$	—
3 Rubanovsky [26]	$A=B=C$	$s_1\gamma_1+s_2\gamma_2+s_3\gamma_3$ $-\frac{1}{2}\frac{abc}{A}\left(\frac{\gamma_1^2}{a}+\frac{\gamma_2^2}{b}+\frac{\gamma_3^2}{c}\right)$ $+\frac{1}{2}n[(b+c)\gamma_1^2+(c+a)\gamma_2^2+(a+b)\gamma_3^2]$	$((a-An)\gamma_1,$ $(b-An)\gamma_2,$ $(c-An)\gamma_3)$	$\frac{1}{2}A[(b+c)\,p^2+(c+a)\,q^2+(a+b)\,r^2]$ $+s_1(Ap+a)\gamma_1+s_2(Aq+b)\gamma_2$ $+s_3(Ar+c)\gamma_3-abc\left(p\frac{\gamma_1}{a}+q\frac{\gamma_2}{b}+r\frac{\gamma_3}{c}\right)$	Lyapunov [27] ($s_1=s_2=s_3=0$)
4 Yehia [5, 19]	$A=B=2C$	$C(a_1\gamma_1+a_2\gamma_2)$	$(0,0,Ck)$	$(p^2-q^2-a_1\gamma_1+a_2\gamma_2)^2$ $+(2pq-a_1\gamma_2-a_2\gamma_1)^2$ $+2k(r-k)(p^2+q^2)-4k\gamma_3(a_1p+a_2q)$	Kovalevskaya ($k=0$) [18]
5 Yehia and Bedweihy [28]	$A=B=2C$	$C\left(a_1\gamma_1+a_2\gamma_2+\frac{\varepsilon}{\sqrt{1-\gamma_3^2}}\right)$	$\mathbf{0}$	$(p^2-q^2-a_1\gamma_1+a_2\gamma_2)^2$ $+(2pq-a_1\gamma_2-a_2\gamma_1)^2$ $+\varepsilon\left[\frac{2(p^2+q^2)}{\sqrt{1-\gamma_3^2}}+\frac{\varepsilon}{1-\gamma_3^2}\right]$	Kovalevskaya ($\varepsilon=0$) [18]
6 Rubanovsky [26]	—	$-n(\boldsymbol{k}\cdot\boldsymbol{\gamma}+\frac{1}{2}a[\mathrm{tr}(\boldsymbol{I}^{-1})-\boldsymbol{\gamma}\boldsymbol{I}^{-1}\cdot\boldsymbol{\gamma}])$ $-\frac{1}{2}n^2\boldsymbol{\gamma}\boldsymbol{I}\cdot\boldsymbol{\gamma}$	$\boldsymbol{k}+a\boldsymbol{\gamma}\boldsymbol{I}^{-1}-2n\boldsymbol{\gamma}\bar{\boldsymbol{I}}$	$\frac{1}{2}\lvert\boldsymbol{\omega}\boldsymbol{I}+\boldsymbol{k}\rvert^2-a\boldsymbol{\omega}\cdot\boldsymbol{\gamma}$	Kharlamov ($k_2=0$) [9] Steklov ($\boldsymbol{k}=\mathbf{0}$) [8] Joukovsky ($a=n=0$) [29] Euler ($\boldsymbol{k}=\mathbf{0}$, $a=n=0$) [11]
7 Langrage's type case [16, 1]	$B=A$	$V(\gamma_3)$	$(-\gamma_1 I_3'(\gamma_3),-\gamma_2 I_3'(\gamma_3),$ $I_3-2\gamma_3\ell+(\gamma_1^2+\gamma_2^2)\ell'(\gamma_3))$	$Cr+I_3$	Clebsch [21], Lagrange [12]

Table 2. Conditional integrable cases on the level $f = 0$.

	Reference	Conditions	V	μ	I_4	Previous cases
1	Yehia [33]	$A = B = 4C$	$C\left(a_1\gamma_1 + a_2\gamma_2 + \frac{\lambda}{\gamma_3^2}\right)$	$(0, 0, Ck)$	$(r-k)\left(p^2+q^2+\frac{\lambda}{2\gamma_3^2}\right) - \gamma_3(a_1 p + a_2 q)$	Sretensky ($\lambda = 0$) [24]
						Goriatchev ($k = 0$) [14] Goriatchev–Chaplygin ($\lambda = k = 0$) (e.g. [10, 17])
2	Yehia [34]	$A = B = 2C$	$C\Big(a_1\gamma_1 + a_2\gamma_2$	$(0, 0, Ck)$	$\left[p^2 - q^2 - a_1\gamma_1 + a_2\gamma_2 + b_1\gamma_3^2 - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{2\gamma_3^2}\right]^2$	Yehia ($\lambda = 0$) [31]
			$+b_1(\gamma_1^2-\gamma_2^2) + 2b_2\gamma_1\gamma_2$		$+\left[2pq - a_1\gamma_2 - a_2\gamma_1 + b_2\gamma_3^2 - \frac{\lambda\gamma_1\gamma_2}{\gamma_3^2}\right]^2$	Goriatchev ($k = 0$) [15]
			$+\frac{\lambda}{2\gamma_3^2}\Big)$		$+k(r-k)[2(p^2+q^2) + \lambda\left(1+\frac{1}{\gamma_3^2}\right)$ $-4k\gamma_3[p(a_1 + b_1\gamma_1 + b_2\gamma_2) + q(a_2 + b_2\gamma_1 - b_1\gamma_2)]$	Chaplygin ($\lambda = k = 0$) [30]
3	Yehia and Bedweihy [28]	$A = B = 2C$	$C\left(a_1\gamma_1 + a_2\gamma_2 + \frac{\varepsilon}{\sqrt{1-\gamma_3^2}} + \frac{\lambda}{2\gamma_3^2}\right)$	**0**	$\left(p^2 - q^2 - a_1\gamma_1 + a_2\gamma_2 - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{2\gamma_3^2}\right)^2$	Subcase of Kovalevskaya's
					$+\left(2pq - a_1\gamma_2 - a_2\gamma_1 - \frac{\lambda\gamma_1\gamma_2}{\gamma_3^2}\right)^2$	($\lambda = \varepsilon = 0$)
					$+\varepsilon\left[\frac{2(p^2+q^2)}{\sqrt{1-\gamma_3^2}} + \frac{\varepsilon}{1-\gamma_3^2} + \frac{\lambda\sqrt{1-\gamma_3^2}}{\gamma_3^2}\right]$	
4	Goriatchev [14]	$A = B = 4C/3$	$\frac{C}{3}\frac{a+b\gamma_1}{\gamma_3^{\frac{2}{3}}}$	**0**	$2r(p^2+q^2) + r^3 - 2b\gamma_3^{\frac{1}{3}}p + r\frac{a+b\gamma_1}{\gamma_3^{\frac{2}{3}}}$	—
5	Separable [16, 35]	—	$\frac{A\gamma_1^2+B\gamma_2^2+C\gamma_3^2}{\sqrt{\beta}}$	**0**	$A^2p^2 + B^2q^2 + C^2r^2$	—
			$[u(\alpha-\sqrt{\beta}) + v(\alpha+\sqrt{\beta})]$[a]		$+\frac{1}{\sqrt{\beta}}[(\alpha-\sqrt{\beta})u(\alpha-\sqrt{\beta}) + (\alpha+\sqrt{\beta})v(\alpha+\sqrt{\beta})]$ [16]	
6	Separable [16, 36, 37]	$A = B = C$	$\frac{[u(\alpha'-\sqrt{\beta'})+v(\alpha'+\sqrt{\beta'})]}{\sqrt{\beta'}}$[b]	**0**	$(ap^2 + bq^2 + cr^2)$	—
					$+\frac{1}{\sqrt{\beta'}}[(\alpha'+\sqrt{\beta'})u(\alpha'-\sqrt{\beta'}) + (\alpha'-\sqrt{\beta'})v(\alpha'+\sqrt{\beta'})]$ [16]	
7	Yehia [38]	$A = B = C$	$-\frac{N^2abc}{2A}\left(\frac{\gamma_1^2}{a} + \frac{\gamma_2^2}{b} + \frac{\gamma_3^2}{c}\right)$	$(a\gamma_1, b\gamma_2, c\gamma_3)$	$\frac{1}{2}A[(b+c)p^2 + (c+a)q^2 + (a+b)r^2]$	Separable ($N = 0$),
			$+\frac{K}{\sqrt{\beta'}}$		$-Nabc\left(p\frac{\gamma_1}{a} + q\frac{\gamma_2}{b} + r\frac{\gamma_3}{c}\right) + \frac{1}{2}K\frac{a+b+c+a\gamma_1^2+b\gamma_2^2+c\gamma_3^2}{\sqrt{\beta'}}$	subcase of Lyapunov's ($K = 0$)
8	Yehia [32]	—	Nv_0^{c}	$\frac{\partial F}{\partial \gamma} - \Phi\gamma^{\text{c}}$	Not constructed in Euler–Poisson variables	—

[a] $\alpha = AB + BC + CA - ABC\left(\frac{\gamma_1^2}{A} + \frac{\gamma_2^2}{B} + \frac{\gamma_3^2}{C}\right)$, $\beta = \alpha^2 - 4ABC(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)$. [b] $\alpha' = a + b + c - (a\gamma_1^2 + b\gamma_2^2 + c\gamma_3^2)$, $\beta' = \alpha^2 - 4abc\left(\frac{\gamma_1^2}{a} + \frac{\gamma_2^2}{b} + \frac{\gamma_3^2}{c}\right)$.

[c] $v_0 = \frac{A(B+C)\gamma_1^2+B(C+A)\gamma_2^2+C(A+B)\gamma_3^2}{A\gamma_1^2+B\gamma_2^2+C\gamma_3^2}$, $F = J\frac{A(B+C)\gamma_1^2+B(C+A)\gamma_2^2+C(A+B)\gamma_3^2}{\sqrt{ABC(A\gamma_1^2+B\gamma_2^2+C\gamma_3^2)}}$, $\Phi = \frac{J}{\sqrt{ABC(A\gamma_1^2+B\gamma_2^2+3C\gamma_3^2)}}\left[2(A+B+C)v_0 - 3v_0^2 + \frac{2ABC}{A\gamma_1^2+B\gamma_2^2+C\gamma_3^2}\right]$.

5. Conclusion

(1) The equations of motion of a rigid body acted upon by potential and gyroscopic forces are shown to be form-invariant under the rotation transformation with position-dependent angular velocity.

(2) Dual interpretation is given to the transformed equations as describing motion of another body relative to an inertial frame under additional active forces. The picture of the latter motion in all its detail is obtained from that of the original one through the given transformation.

(3) This, in particular, enables the construction of:

   (a) Five general integrable cases generalizing known ones by including several parameters.

   (b) Conditional generalizations, involving an arbitrary function of the position in the structure of forces acting on the body, for all the fifteen known (general and conditional) integrable cases as well as for all particular solutions of the problem.

   Detailed examples of each type are given.

(4) Detailed physical interpretation is given for two of the new general integrable cases as describing motion of a magnetized and electrically charged body under the action of gravitational, electric, magnetic and Lorentz forces. A generalization of Grioli's precession describes a problem of motion of a body in a liquid.

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